

# ON ASSOCIATED GRADED MODULES HAVING A PURE RESOLUTION

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**ABSTRACT.** Let  $A = K[[X_1, \dots, X_n]]$  and let  $\mathfrak{m} = (X_1, \dots, X_n)$ . Let  $M$  be a Cohen-Macaulay  $A$ -module of codimension  $p$ . In this paper we give a necessary and sufficient condition for the associated graded module  $G_{\mathfrak{m}}(M)$  to have a pure resolution over the polynomial ring  $G_{\mathfrak{m}}(A) \cong K[X_1, \dots, X_n]$ .

## 1. INTRODUCTION

Let  $R = K[X_1, \dots, X_n]$  and let  $M$  be a finitely generated graded  $R$ -module of projective dimension  $p$ . Recall that  $M$  has a *pure resolution* of type  $(d_0, d_1, \dots, d_p)$  if the minimal resolution of  $M$  is of the following form:

$$0 \rightarrow R(-d_p)^{\beta_p} \rightarrow \dots \rightarrow R(-d_2)^{\beta_2} \rightarrow R(-d_1)^{\beta_1} \rightarrow R(-d_0)^{\beta_0} \rightarrow 0.$$

Herzog and Kühn showed in [3] that the Betti numbers of a pure resolution of a Cohen-Macaulay algebra are determined by its type and Huneke and Miller computed in [4] the multiplicity of such an algebra, also in terms of its type. These two results can be extended to Cohen-Macaulay  $R$ -modules, see [1, p. 88]. In the beautiful paper [1], Boig and Söderberg conjectured that the Betti diagram of any Cohen-Macaulay  $R$ -module is a non-negative linear combination of pure diagrams; furthermore, any pure diagram is a rational multiple of the Betti diagram of some Cohen-Macaulay  $R$ -module. This conjecture was proved in [2].

Let  $A = K[[X_1, \dots, X_n]]$  and let  $\mathfrak{m} = (X_1, \dots, X_n)$ . Let  $G_{\mathfrak{m}}(A)$  be the associated graded ring of  $A$  with respect to  $\mathfrak{m}$ , i.e.,  $G_{\mathfrak{m}}(A) = \bigoplus_{n \geq 0} \mathfrak{m}^n / \mathfrak{m}^{n+1}$ . It is well-known that  $G_{\mathfrak{m}}(A) \cong K[X_1, \dots, X_n]$ . Let  $M$  be a finitely generated  $A$ -module and let  $G_{\mathfrak{m}}(M) = \bigoplus_{n \geq 0} \mathfrak{m}^n M / \mathfrak{m}^{n+1} M$  be the associated graded module of  $M$  with respect to  $\mathfrak{m}$ . A natural question is when does  $G_{\mathfrak{m}}(M)$  have a pure resolution? To answer this question we construct a pure complex attached to  $M$  as follows:

**Construction 1.1.** (1). Let  $\phi: A^n \rightarrow A^m$  be a non-zero  $A$ -linear map. Assume  $\text{image } \phi \subseteq \mathfrak{m}^s A^m$  but  $\text{image } \phi \not\subseteq \mathfrak{m}^{s+1} A^m$  for some  $s \geq 0$ . Let  $\phi = (a_{ij})$  where  $a_{ij} \in A$ . By assumption  $a_{ij} \in \mathfrak{m}^s$ . It follows that

$$\phi = \sum_{j \geq s} \phi_j$$

where  $\phi_j$  is a matrix with entries homogeneous forms of degree  $j$ . Set  $\text{in}(\phi) = \phi_s$ . We call  $\text{in}(\phi)$  to be the *initial form* of  $\phi$ . Set  $v(\phi) = s$ ; the *order* of  $\phi$ .

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**Example 1.2.** Let

$$\phi = \begin{pmatrix} X_1^2 + X_2^2 X_3 & 0 & X_3^2 \\ 0 & X_2^4 & X_1 X_3 + X_2^3 \\ X_1^3 & 0 & X_1 X_2 X_3 + X_2^4 \end{pmatrix}.$$

Then

$$\text{in}(\phi) = \begin{pmatrix} X_1^2 & 0 & X_3^2 \\ 0 & 0 & X_1 X_3 \\ 0 & 0 & 0 \end{pmatrix}.$$

Also  $v(\phi) = 2$ .

(2). Let

$$\mathbb{F}: 0 \rightarrow F_p \xrightarrow{\phi_p} F_{p-1} \rightarrow \cdots \rightarrow F_2 \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \rightarrow 0,$$

be a minimal resolution of  $M$ . Set  $c_i = v(\phi_i)$  for  $i = 1, 2, \dots, p$ . Let  $d_i = \sum_{j=1}^i c_j$  for  $i = 1, 2, \dots, p$ . Let  $\beta_i = \text{rank } F_i$  be the  $i^{\text{th}}$  betti-number of  $M$ . It is easily shown (see 2.3) that we have a complex,  $\text{in}(\mathbb{F})$ ,

$$0 \rightarrow R(-d_p)^{\beta_p} \xrightarrow{\text{in}(\phi_p)} R(-d_{p-1})^{\beta_{p-1}} \rightarrow \cdots \rightarrow R(-d_1)^{\beta_1} \xrightarrow{\text{in}(\phi_1)} R^{\beta_0} \rightarrow 0.$$

We also have an augmentation map  $\epsilon: \text{in}(\mathbb{F}) \rightarrow G_{\mathfrak{m}}(M)$ , i.e., an  $R$ -linear map  $\epsilon: R^{\beta_0} \rightarrow G_{\mathfrak{m}}(M)$  such that  $\epsilon \circ \text{in}(\phi_1) = 0$ . Furthermore  $\epsilon$  is surjective. If  $\mathbb{G}$  is another minimal resolution of  $M$  then it can be easily shown that  $\text{in}(\mathbb{F}) \cong \text{in}(\mathbb{G})$  as augmented complexes, see 2.3.

Our first result is:

**Theorem 1.3.** *Let  $M$  be a finitely generated  $A$ -module. Assume  $G_{\mathfrak{m}}(M)$  has a pure resolution. Let  $\mathbb{F}$  be a minimal free resolution of  $M$ . Then  $\text{in}(\mathbb{F})$  is a minimal free resolution of  $G_{\mathfrak{m}}(M)$ .*

Theorem 1.3 gives a necessary condition for  $G_{\mathfrak{m}}(M)$  to have a pure resolution. The following result gives a sufficient condition for  $G_{\mathfrak{m}}(M)$  to have a pure resolution.

**Theorem 1.4.** *Let  $M$  be a Cohen-Macaulay  $A$ -module and let  $p = \text{projdim } M$ . Let  $\beta_i = \beta_i(M)$ . Let  $\mathbb{F}$  be a minimal resolution of  $M$ . The following conditions are equivalent*

- (i)  $G_{\mathfrak{m}}(M)$  has a pure resolution.
- (ii) The following hold
  - (a)  $\text{in}(\mathbb{F})$  is acyclic.
  - (b) For  $i \geq 1$ ,

$$\beta_i = (-1)^{i+1} \beta_0 \prod_{\substack{1 \leq j \leq p \\ j \neq i}} \frac{d_j}{d_j - d_i}.$$

- (c) The multiplicity of  $M$ ,

$$e_0(M) = \frac{\beta_0}{p!} \prod_{i=1}^p d_i.$$

If  $M = A/I$  then using a computer algebra program one can find  $G_{\mathfrak{m}}(M)$ . However if  $M$  is not a cyclic module there is no computer algebra program to find a presentation of  $G_{\mathfrak{m}}(M)$  as a  $R$ -module. My motivation for this work was to find non-trivial examples of a minimal resolution of  $G_{\mathfrak{m}}(M)$ . Even when  $p = \text{projdim } M = 1, 2$  this is a non-trivial problem even when  $G_{\mathfrak{m}}(M)$  is Cohen-Macaulay.

Here is an overview of the contents of the paper. In section two we give our construction of the complex  $\text{in}(\mathbb{F})$ . In section three we prove Theorem 1.3. In the next section we prove Theorem 1.4. Finally in section five we give an non-trivial example of  $G_{\mathfrak{m}}(M)$  having a pure resolution.

## 2. CONSTRUCTION OF THE COMPLEX $\text{in}(\mathbb{F})$

Throughout  $A = k[[X_1, \dots, X_n]]$  and let  $\mathfrak{m}$  be the maximal ideal of  $A$ . All  $A$ -modules considered will be finitely generated. Let  $R = G_{\mathfrak{m}}(A) = \bigoplus_{n \geq 0} \mathfrak{m}^n / \mathfrak{m}^{n+1}$  be the associated graded ring of  $A$ . It is well-known that  $R \cong k[X_1, \dots, X_n]$ . Let  $M$  be an  $A$ -module and let  $G_{\mathfrak{m}}(M) = \bigoplus_{n \geq 0} \mathfrak{m}^n M / \mathfrak{m}^{n+1} M$  be the associated graded module of  $M$ . Clearly  $G_{\mathfrak{m}}(M)$  is a finitely generated  $R$ -module. Let  $\mathbb{F}$  be a minimal resolution of  $M$ . In this section we construct our complex  $\text{in}(\mathbb{F})$  of  $R$ -modules. We also show that there is an augmentation  $\epsilon: \text{in}(\mathbb{F}) \rightarrow G_{\mathfrak{m}}(M)$  with  $\epsilon$  surjective.

**Proposition 2.1.** *Let  $F_2 \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0$  be a complex of free  $A$ -modules. Let  $v(\phi_1) = c_1$  and  $v(\phi_2) = c_2$ . Set  $d_1 = c_1$  and  $d_2 = c_1 + c_2$ . Then*

(1) *we have a complex of  $R = G_{\mathfrak{m}}(A)$ -modules*

$$G_{\mathfrak{m}}(F_2)(-d_2) \xrightarrow{\text{in}(\phi_2)} G_{\mathfrak{m}}(F_1)(-d_1) \xrightarrow{\text{in}(\phi_1)} G_{\mathfrak{m}}(F_0).$$

(2) *Let  $M = \text{coker } \phi_1$ . Consider the natural map  $\epsilon: G_{\mathfrak{m}}(F_0) \rightarrow G_{\mathfrak{m}}(M)$ . Then*

- (a)  $\epsilon \circ \text{in}(\phi_1) = 0$ .
- (b)  $\epsilon$  is surjective.

*Proof.* If  $E$  is an  $A$ -module then set  $\mathfrak{m}^i E = E$  for  $i \leq 0$ . Notice that for all  $i \in \mathbb{Z}$  we have an complex

$$\mathfrak{m}^{i-d_2} F_2 \rightarrow \mathfrak{m}^{i-d_1} F_1 \rightarrow \mathfrak{m}^i F_0 \rightarrow \mathfrak{m}^i M \rightarrow 0.$$

After tensoring with  $A/\mathfrak{m}$  and collecting terms we have a complex

$$G_{\mathfrak{m}}(F_2)(-d_2) \xrightarrow{\alpha} G_{\mathfrak{m}}(F_1)(-d_1) \xrightarrow{\beta} G_{\mathfrak{m}}(F_0) \xrightarrow{\epsilon} G_{\mathfrak{m}}(M) \rightarrow 0.$$

Observe that  $\alpha = \text{in}(\phi_2)$  and  $\beta = \text{in}(\phi_1)$ . Also clearly  $\epsilon$  is surjective.  $\square$

Next we show

**Proposition 2.2.** *Let  $\mathbb{F}: F_2 \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0$  and  $\mathbb{G}: G_2 \xrightarrow{\psi_2} G_1 \xrightarrow{\psi_1} G_0$  be complexes of free  $A$ -modules. Suppose we have a commutative diagram*

$$\begin{array}{ccccc} F_2 & \xrightarrow{\phi_2} & F_1 & \xrightarrow{\phi_1} & F_0 \\ \downarrow \theta_2 & & \downarrow \theta_1 & & \downarrow \theta_0 \\ G_2 & \xrightarrow{\psi_2} & G_1 & \xrightarrow{\psi_1} & G_0 \end{array}$$

*such that  $\theta_i$  are isomorphism's for  $i = 0, 1, 2$ . Then*

(1)  $v(\phi_i) = v(\psi_i)$  for  $i = 1, 2$ .

- (2) Let  $v(\phi_1) = c_1$  and  $v(\phi_2) = c_2$ . Set  $d_1 = c_1$  and  $d_2 = c_1 + c_2$ . Then we have a commutative diagram of  $R = G_{\mathfrak{m}}(A)$ -modules

$$\begin{array}{ccccc}
 G_{\mathfrak{m}}(F_2)(-d_2) & \xrightarrow{\text{in}(\phi_2)} & G_{\mathfrak{m}}(F_1)(-d_1) & \xrightarrow{\text{in}(\phi_1)} & G_{\mathfrak{m}}(F_0) \\
 \downarrow \text{in}(\theta_2) & & \downarrow \text{in}(\theta_1) & & \downarrow \text{in}(\theta_0) \\
 G_{\mathfrak{m}}(G_2)(-d_2) & \xrightarrow{\text{in}(\psi_2)} & G_{\mathfrak{m}}(G_1)(-d_1) & \xrightarrow{\text{in}(\psi_1)} & G_{\mathfrak{m}}(G_0)
 \end{array}$$

such that

- (a) The rows are complexes of  $R$ -modules.  
(b)  $\text{in}(\theta_i)$  is an isomorphism for  $i = 0, 1, 2$ .  
(3) Set  $M = \text{coker } \phi_1$  and  $M' = \text{coker } \phi_2$  and let  $\xi: M \rightarrow M'$  be the isomorphism induced by the above commutative diagram. Then we have a commutative diagram

$$\begin{array}{ccccc}
 G_{\mathfrak{m}}(F_0) & \xrightarrow{\epsilon} & G_{\mathfrak{m}}(M) & \longrightarrow & 0 \\
 \downarrow \text{in}(\theta_0) & & \downarrow G_{\mathfrak{m}}(\xi) & & \\
 G_{\mathfrak{m}}(G_0) & \xrightarrow{\epsilon'} & G_{\mathfrak{m}}(M') & \longrightarrow & 0
 \end{array}$$

*Proof.* (0) Let  $\delta: A^m \rightarrow A^m$  be invertible  $A$ -linear map. We write  $\delta = \sum_{j \geq 0} \delta_j$  where  $\delta_j$  is a  $m \times m$  matrix of forms of degree  $j$ . As  $\delta \otimes A/\mathfrak{m}: k^m \rightarrow k^m$  is an isomorphism we get that  $\delta_0$  is an invertible matrix. Also notice  $\delta_0 = \text{in}(\delta)$ .

(1) and (2): Let  $r = v(\phi_1)$ . We write  $\phi_1 = \sum_{j \geq r} \phi_{1,j}$  where  $\phi_{1,j}$  is a matrix of forms of degree  $j$ . Let  $s = v(\psi_1)$ . Write  $\psi_1 = \sum_{j \geq s} \psi_{1,j}$  where  $\psi_{1,j}$  is a matrix of forms of degree  $j$ . For  $i = 0, 1$  write  $\theta_i = \sum_{j \geq 0} \theta_{i,j}$  as before. As  $\theta_0 \circ \phi_1 = \psi_1 \circ \theta_1$  we get that  $\theta_{0,0} \circ \phi_{1,r} = \psi_{1,s} \circ \theta_{1,0}$ . By (0) we get that  $\theta_{0,0}$  and  $\theta_{1,0}$  are invertible matrices of constants. So  $r = s$ . We also get  $\text{in}(\theta_0) \circ \text{in}(\phi_1) = \text{in}(\psi_1) \circ \text{in}(\theta_1)$ . Similarly we get  $v(\phi_2) = v(\psi_2)$  and  $\text{in}(\theta_1) \circ \text{in}(\phi_2) = \text{in}(\psi_2) \circ \text{in}(\theta_2)$ .

2(a) This follows from Proposition 2.1.

2(b) This follows from (0).

(3) This is obvious. □

**Construction 2.3.** Let

$$\mathbb{F}: 0 \rightarrow F_p \xrightarrow{\phi_p} F_{p-1} \rightarrow \cdots \rightarrow F_2 \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \rightarrow 0,$$

be a minimal resolution of  $M$ . Set  $c_i = v(\phi_i)$  for  $i = 1, 2, \dots, p$ . Let  $d_i = \sum_{j=1}^i c_j$  for  $i = 1, 2, \dots, p$ . Let  $\beta_i = \text{rank } F_i$  be the  $i^{\text{th}}$  betti-number of  $M$ . Note  $R_i^\beta = G_{\mathfrak{m}}(F_i)$ . By Proposition 2.1 it follows that we have a complex,  $\text{in}(\mathbb{F})$ ,

$$0 \rightarrow R(-d_p)^{\beta_p} \xrightarrow{\text{in}(\phi_p)} R(-d_{p-1})^{\beta_{p-1}} \rightarrow \cdots \rightarrow R(-d_1)^{\beta_1} \xrightarrow{\text{in}(\phi_1)} R^{\beta_0} \rightarrow 0.$$

By Proposition 2.1 we also have an augmentation map  $\epsilon: \text{in}(\mathbb{F}) \rightarrow G_{\mathfrak{m}}(M)$ , i.e., an  $R$ -linear map  $\epsilon: R^{\beta_0} \rightarrow G_{\mathfrak{m}}(M)$  such that  $\epsilon \circ \text{in}(\phi_1) = 0$ . Furthermore  $\epsilon$  is clearly surjective. If  $\mathbb{G}$  is another minimal resolution of  $M$  then by Proposition 2.2 it follows that we have an isomorphism of augmented complexes  $\text{in}(\mathbb{F})$  and  $\text{in}(\mathbb{G})$ .

## 3. PROOF OF THEOREM 1.3

In this section we give a proof of Theorem 1.3. We first prove:

**Lemma 3.1.** *Let  $F_1 \xrightarrow{\phi} F_0 \rightarrow M \rightarrow 0$  be part of a minimal resolution of  $M$ . Assume that the minimal resolution of  $G_{\mathfrak{m}}(M)$  has the following form  $\cdots \rightarrow R^a(-s) \rightarrow G_{\mathfrak{m}}(F_0) \rightarrow G_{\mathfrak{m}}(M) \rightarrow 0$ . Set  $N = \text{image } \phi$  and  $\mathcal{F} = \{N_i = \mathfrak{m}^i F_0 \cap N\}_{i \in \mathbb{Z}}$ . Then*

- (1)  $N_i = \mathfrak{m}^{i-s} N$  for all  $i \geq 0$ .
- (2)  $\text{rank } F_1 = a$ .
- (3)  $v(\phi) = s$ .
- (4) *The sequence  $G_{\mathfrak{m}}(F_1)(-s) \xrightarrow{\text{in}(\phi)} G_{\mathfrak{m}}(F_0) \rightarrow G_{\mathfrak{m}}(M) \rightarrow 0$  can be extended to a minimal resolution of  $G_{\mathfrak{m}}(M)$ .*
- (5)  $\text{image in}(\phi) \cong G_{\mathfrak{m}}(N)(-s)$ .

*Proof.* (1) It is well-known that  $\mathcal{F}$  is an  $\mathfrak{m}$ -stable filtration on  $N$ . Furthermore we have an exact sequence

$$(*) \quad 0 \rightarrow G_{\mathcal{F}}(N) \rightarrow G_{\mathfrak{m}}(F_0) \rightarrow G_{\mathfrak{m}}(M) \rightarrow 0.$$

By our assumption it follows that  $G_{\mathcal{F}}(N)$  is generated in degree  $s$ . So we have  $N_s = N$  and  $N_{s+j} = \mathfrak{m}^j N + N_{s+j+1}$  for all  $j \geq 1$ . As  $\mathcal{F}$  is  $\mathfrak{m}$ -stable there exists  $j_0$  such that  $N_{s+j+1} = \mathfrak{m} N_{s+j}$  for all  $j \geq j_0$ . Fix  $j \geq j_0$ . Then  $N_{s+j} = \mathfrak{m}^j N + N_{s+j+1} = \mathfrak{m}^j N + \mathfrak{m} N_{s+j}$ . So by Nakayama Lemma  $N_{s+j} = \mathfrak{m}^j N$ . We now show by descending induction that  $N_{s+j} = \mathfrak{m}^j N$  for all  $j \leq j_0$ . This is true for  $j = j_0$  by the previous argument. Assume  $N_{s+j+1} = \mathfrak{m}^{j+1} N$  for some  $j \leq j_0 - 1$ . Then notice

$$\mathfrak{m}^{j+1} N \subseteq \mathfrak{m} N_{s+j} \subseteq N_{s+j+1} = \mathfrak{m}^{j+1} N.$$

So we have  $N_{s+j+1} = \mathfrak{m} N_{s+j}$ . As  $N_{s+j} = \mathfrak{m}^j N + N_{s+j+1} = \mathfrak{m}^j N + \mathfrak{m} N_{s+j}$ , by Nakayama Lemma we get that  $N_{s+j} = \mathfrak{m}^j N$ .

(2) As  $R^a(-s) \rightarrow G_{\mathfrak{m}}(N)(-s) \rightarrow 0$  is minimal we have  $a = \mu(G_{\mathfrak{m}}(N)) = \mu(N) = \text{rank } F_1$ .

(3) Set  $r = v(\phi)$ . By 2.1 we have a complex

$$G_{\mathfrak{m}}(F_1)(-r) \xrightarrow{\text{in}(\phi)} G_{\mathfrak{m}}(F_0) \xrightarrow{\epsilon} G_{\mathfrak{m}}(M) \rightarrow 0.$$

So  $\ker \epsilon$  contains an element of degree  $r$ . So  $s \leq r$ . Furthermore note that  $N \subseteq \mathfrak{m}^r F_0$ . So  $N_j = \mathfrak{m}^j F_0 \cap N = N$  for  $j \leq r$ . It follows that  $s \geq r$ . Thus  $s = r$ .

(4) We first show that the complex is exact.

$$(**) \quad \mathfrak{m}^{i-s} F_1 \xrightarrow{\phi_{i-s}} \mathfrak{m}^i F_1 \xrightarrow{\epsilon_i} \mathfrak{m}^i M \rightarrow 0 \quad \text{is exact for all } i \geq 0.$$

Here  $\phi_{i-s}$  is the restriction of  $\phi$  to  $\mathfrak{m}^{i-s} F_1$ . Notice  $\ker \epsilon_i = N \cap \mathfrak{m}^i F = N_i = \mathfrak{m}^{i-s} N$  and the map  $\phi_{i-s}$  maps  $\mathfrak{m}^{i-s} F_1$  surjectively to  $\mathfrak{m}^{i-s} N$ . We now tensor the exact sequence  $(**)$  with  $A/\mathfrak{m}$  to get the exact sequence

$$\frac{\mathfrak{m}^{i-s} F_1}{\mathfrak{m}^{i+1-s} F_1} \xrightarrow{\overline{\phi_{i-s}}} \frac{\mathfrak{m}^i F_0}{\mathfrak{m}^{i+1} F_0} \rightarrow \frac{\mathfrak{m}^i M}{\mathfrak{m}^{i+1} M} \rightarrow 0,$$

for all  $i \geq 0$ . Thus we have an exact sequence

$$(\dagger) \quad G_{\mathfrak{m}}(F_1)(-s) \xrightarrow{\overline{\phi}} G_{\mathfrak{m}}(F_0) \rightarrow G_{\mathfrak{m}}(M) \rightarrow 0.$$

It can be easily verified that  $\overline{\phi} = \text{in}(\phi)$ . Furthermore as  $G_{\mathfrak{m}}(F_1) = R^a$  we get that  $(\dagger)$  is part of a minimal resolution of  $G_{\mathfrak{m}}(M)$ .

(5) This follows from the exact sequence  $(*)$  and (1).  $\square$

We now give

*Proof of Theorem 1.3.* We prove the result by induction on  $p = \text{projdim } M$ . The result clearly holds when  $p = 0$ . Furthermore when  $p = 1$  the result follows from Lemma 3.1. We assume the result when  $p = r \geq 1$  and prove it when  $p = r + 1$ . Set  $N = \text{Syz}_1^A(M)$  and let  $v(\phi_1) = s_1$ . Then by Lemma 3.1 we get

$$\text{Syz}_1^R(G_{\mathfrak{m}}(M)) = G_{\mathfrak{m}}(N)(-s_1).$$

It follows that  $G_{\mathfrak{m}}(N)$  has a pure resolution. Truncate  $\mathbb{F}$  to obtain a minimal resolution  $\mathbb{F}'$  of  $N$ . By induction hypotheses  $\text{in}(\mathbb{F}')$  is a minimal resolution of  $G_{\mathfrak{m}}(N)$ . After shifting  $\text{in}(\mathbb{F}')$  suitably and by Lemma 3.1 we get that  $\text{in}(\mathbb{F})$  is a minimal resolution of  $G_{\mathfrak{m}}(M)$ .  $\square$

#### 4. PROOF OF THEOREM 1.4

In this section we give a proof of our main result. We need the following result due to Herzog-Kühl (see [3, Theorem 1]) regarding modules having pure resolutions.

**Lemma 4.1.** *Let  $R = k[X_1, X_2, \dots, X_n]$  and suppose  $E$  is a finitely generated graded  $R$ -module with pure resolution of type  $(0, d_1, d_2, \dots, d_p)$ . Set  $\beta_i = \beta_i(E)$ . If for  $i \geq 1$*

$$\beta_i = (-1)^{i+1} \beta_0 \prod_{j \neq i} \frac{d_j}{d_j - d_i},$$

*then  $E$  is a Cohen-Macaulay  $R$ -module.*

We now give a proof of our main result:

*Proof of Theorem 1.4.* If  $G_{\mathfrak{m}}(M)$  has a pure resolution then by Theorem 1.3 the complex  $\text{in}(\mathbb{F})$  is a minimal resolution of  $G_{\mathfrak{m}}(M)$ . The result now follow from [1, p. 88].

Conversely if  $\text{in}(\mathbb{F})$  is acyclic and the betti -numbers satisfy the Herzog-Kühl conditions then by Lemma 4.1,  $E = \text{coker } \text{in}(\phi_1)$  is Cohen-Macaulay of dimension  $n - p = \dim M$ . Note we also have a surjective homomorphism  $\epsilon: G_{\mathfrak{m}}(F_0) \rightarrow G_{\mathfrak{m}}(M)$  with  $\epsilon \circ \text{in}(\phi_1) = 0$ . So we have an exact sequence

$$0 \rightarrow K \rightarrow E \rightarrow G_{\mathfrak{m}}(M) \rightarrow 0.$$

Note  $\dim K \leq \dim E = \dim G_{\mathfrak{m}}(M)$ . As multiplicity of  $E$  equals multiplicity of  $G_{\mathfrak{m}}(M)$  it follows that  $\dim K < \dim E$ . But  $E$  is Cohen-Macaulay. Therefore  $K = 0$ . So  $G_{\mathfrak{m}}(M) = E$  has a pure resolution.  $\square$

#### 5. AN EXAMPLE

All the computations in this section were done using the computer algebra package SINGULAR. Let  $Q = k[x, y, z]_{\mathfrak{m}}$  where  $\mathfrak{m} = (x, y, z)$ . Also  $A = \widehat{Q}$  and  $R = G_{\mathfrak{m}}(A) = G_{\mathfrak{m}}(Q) = k[x, y, z]$ . Using SINGULAR one can verify that we have an exact sequence

$$0 \rightarrow Q^2 \xrightarrow{\psi} Q^4 \xrightarrow{\phi} Q^2 \rightarrow M \rightarrow 0,$$

where

$$\phi = \begin{pmatrix} y^2 & x^2 & z^3 & 0 \\ 0 & 0 & z^2 & x^2 \end{pmatrix} \quad \text{and} \quad \psi = \begin{pmatrix} -x^2 & 0 \\ y^2 & z^3 \\ 0 & -x^2 \\ 0 & z^2 \end{pmatrix}.$$

Notice that the zeroth Fitting ideal of  $M$  is

$$\text{Fitt}_0(M) = I_2(\phi) = (y^2 z^2, y^2 x^2, x^2 z^2, x^4, z^3 x^2).$$

It is well-known that  $\text{Fitt}_0(M) \subseteq \text{ann}(M)$  and that  $\sqrt{\text{Fitt}_0(M)} = \sqrt{\text{ann}(M)}$ . If  $P$  is a prime containing  $\text{Fitt}_0(M)$  then clearly  $P \supseteq (x, y)$  or  $P \supseteq (x, z)$ . Thus  $\dim M = 1$ . Therefore  $M$  is Cohen-Macaulay.

Notice

$$\text{in}(\phi) = \begin{pmatrix} y^2 & x^2 & 0 & 0 \\ 0 & 0 & z^2 & x^2 \end{pmatrix} \quad \text{and} \quad \text{in}(\psi) = \begin{pmatrix} -x^2 & 0 \\ y^2 & 0 \\ 0 & -x^2 \\ 0 & z^2 \end{pmatrix}.$$

Using SINGULAR or the Buchsbaum-Eisenbud criterion it follows that the complex

$$(*) \quad 0 \rightarrow R(-4)^2 \xrightarrow{\text{in}(\psi)} R(-2)^4 \xrightarrow{\text{in}(\phi)} R^2 \rightarrow 0$$

is acyclic.

Set  $T = Q/(x^4, y^2 z^2)$ . Then  $M$  is a Cohen-Macaulay  $T$ -module. Also note that  $T$  is Cohen-Macaulay of dimension one with  $e(T) = 16$ . We have an exact sequence

$$T^4 \xrightarrow{\bar{\phi}} T^2 \rightarrow M \rightarrow 0.$$

Let  $N = \text{Syz}_1^T(M)$ . Then using SINGULAR one can check that  $e(N) = 24$ . Thus  $e(M) = 8$ . Therefore by Theorem 1.4 we get that  $(*)$  is a minimal resolution of  $G_{\mathfrak{m}}(M)$ .

**Remark 5.1.** Although in this example  $p = 2$  and  $\dim M = 1$ , this is the first non-trivial example where we have an explicit resolution of  $G_{\mathfrak{m}}(M)$ .

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